# A note on mean-value properties of harmonic functions on the hypercube

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**Abstract**

For functions defined on the $n$-dimensional hypercube $I_n(r) = \{\boldsymbol{x} \in \mathbb{R}^n \mid |x_i| \leq r,\ i = 1, 2, \ldots, n\}$ and harmonic therein, we establish certain analogues of Gauss surface and volume mean-value formulas for harmonic functions on the ball in $\mathbb{R}^n$ and their extensions for polyharmonic functions. The relation of these formulas to best one-sided $L^1$-approximation by harmonic functions on $I_n(r)$ is also discussed.

## 1. Introduction

This note is devoted to formulas for calculation of integrals over the $n$-dimensional hypercube centered at $\mathbf{0}$

$$I_n := I_n(r) := \{\boldsymbol{x} \in \mathbb{R}^n \mid |x_i| \leq r,\ i = 1, 2, \ldots, n\},\ r > 0,$$

and its boundary $P_n := P_n(r) := \partial I_n(r)$, based on integration over hyperplanar subsets of $I_n$ and exact for harmonic or polyharmonic functions. They are presented in Section 2 and can be considered as natural analogues on $I_n$ of Gauss surface and volume mean-value formulas for harmonic functions ([5]) and Pizzetti formula [8],[3, Part IV, Ch. 3, pp. 287-288] for polyharmonic functions on the ball in $\mathbb{R}^n$. Section 3 deals with the best one-sided $L^1$-approximation by harmonic functions.

Let us remind that a real-valued function $f$ is said to be *harmonic* (*polyharmonic of degree* $m \geq 2$) in a given domain $\Omega \subset \mathbb{R}^n$ if $f \in C^2(\Omega)$

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$(f \in C^{2m}(\Omega))$ and $\Delta f = 0$ ($\Delta^m f = 0$) on $\Omega$, where $\Delta$ is the Laplace operator and $\Delta^m$ is its $m$-th iterate

$$\Delta f := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}, \quad \Delta^m f := \Delta(\Delta^{m-1} f).$$

For any set $D \subset \mathbb{R}^n$ , denote by $\mathcal{H}(D)$ ($\mathcal{H}^m(D), m \geq 2$) the linear space of all functions that are harmonic (polyharmonic of degree $m$) in a domain containing $D$. The notation $d\lambda_n$ will stand for the Lebesgue measure in $\mathbb{R}^n$.

## 2. Mean-value theorems

Let $B_n(r) := \{\boldsymbol{x} \in \mathbb{R}^n \mid \|x\| := \left(\sum_{i=1}^n x_i^2\right)^{1/2} \leq r\}$ and $S_n(r) := \{\boldsymbol{x} \in \mathbb{R}_n \mid \|x\| = r\}$ be the ball and the hypersphere in $\mathbb{R}^n$ with center $\mathbf{0}$ and radius $r$. The following famous formulas are basic tools in harmonic function theory and state that for any function $h$ which is harmonic on $B_n(r)$ both the average over $S_n(r)$ and the average over $B_n(r)$ are equal to $h(\mathbf{0})$.

**The surface mean-value theorem.** *If $h \in \mathcal{H}(B_n(r))$, then*

$$\frac{1}{\sigma_{n-1}(S_n(r))} \int_{S_n(r)} h \, d\sigma_{n-1} = h(\mathbf{0}), \tag{1}$$

*where $d\sigma_{n-1}$ is the $(n-1)$-dimensional surface measure on the hypersphere $S_n(r)$.*

**The volume mean-value theorem.** *If $h \in \mathcal{H}(B_n(r))$, then*

$$\frac{1}{\lambda_n(B_n(r))} \int_{B_n(r)} h \, d\lambda_n = h(\mathbf{0}). \tag{2}$$

The balls are known to be the only sets in $\mathbb{R}^n$ satisfying the surface or the volume mean-value theorem. This means that if $\Omega \subset \mathbb{R}^n$ is a nonvoid domain with a finite Lebesgue measure and if there exists a point $\boldsymbol{x}_0 \in \Omega$ such that $h(\boldsymbol{x}_0) = \frac{1}{\lambda_n(\Omega)} \int_\Omega h \, d\lambda_n$ for every function $h$ which is harmonic and integrable on $\Omega$, then $\Omega$ is an open ball centered at $\boldsymbol{x}_0$ (see [6]). The mean-value properties can also be reformulated in terms of quadrature domains [9]. Recall that $\Omega \subset \mathbb{R}^n$ is said to be a quadrature domain for $\mathcal{H}(\Omega)$, if $\Omega$ is a connected open set and there is a Borel measure $d\mu$ with a compact support $K_\mu \subset \Omega$ such

that $\int_{\bar{\Omega}} f\, d\lambda_n = \int_{K_\mu} f\, d\mu$ for every $\lambda_n$-integrable harmonic function $f$ on $\Omega$. Using the concept of quadrature domains, the volume mean-value property is equivalent to the statement that any open ball in $\mathbb{R}^n$ is a quadrature domain and the measure $d\mu$ is the Dirac measure supported at its center. On the other hand, no domains having "corners" are quadrature domains [7]. From this point of view, the open hypercube $I_n^\circ$ is not a quadrature domain. Nevertheless, here we prove that the closed hypercube $I_n$ is a quadrature set in an extended sense - there exists a measure $d\mu$ with a compact support $K_\mu$ having the above property with $\Omega$ replaced by $I_n$ but the condition $K_\mu \subset I_n^\circ$ is violated exactly at the "corners" (Theorem 1). This property of $I_n$ is of crucial importance for the best one-sided $L^1$-approximation with respect to $\mathcal{H}(I_n)$ (Section 3).

Let us denote by $D_n^{ij}$ the $(n-1)$-dimensional hyperplanar segments of $I_n$ defined by

$$D_n^{ij} := D_n^{ij}(r) := \{\boldsymbol{x} \in I_n \mid |x_k| \leq |x_i| = |x_j|,\ k \neq i, j\}, \quad 1 \leq i < j \leq n.$$

Denote also

$$\omega_k(\boldsymbol{x}) := \frac{(r - \max\{|x_1|, |x_2|, \ldots, |x_n|\})^k}{k!}, \quad k \geq 0,$$

and $d\lambda_m^{\omega_k} := \omega_k\, d\lambda_m$. It can be calculated that

$$\lambda_n^{\omega_k}(I_n) = 2^n n! \frac{r^{n+k}}{(n+k)!}, \quad \lambda_{n-1}^{\omega_k}(P_n) = 2^n n! \frac{r^{n+k-1}}{(n+k-1)!},$$

and

$$\lambda_{n-1}^{\omega_k}(D_n) = 2^{n-1} n! \frac{r^{n+k-1}}{(n+k-1)!}, \quad \text{where} \quad D_n := \cup_{1 \leq i < j \leq n} D_n^{ij}.$$

The following holds true.

**Theorem 1.** *If $h \in \mathcal{H}(I_n)$, then $h$ satisfies:*
*(i)* **Surface mean-value formula for the hypercube**

$$\frac{1}{\lambda_{n-1}(P_n)} \int_{P_n} h\, d\lambda_{n-1} = \frac{1}{\lambda_{n-1}(D_n)} \int_{D_n} h\, d\lambda_{n-1}, \tag{3}$$

*(ii)* **Volume mean-value formula for the hypercube**

$$\frac{1}{\lambda_n^{\omega_k}(I_n)} \int_{I_n} h\, d\lambda_n^{\omega_k} = \frac{1}{\lambda_{n-1}^{\omega_{k+1}}(D_n)} \int_{D_n} h\, d\lambda_{n-1}^{\omega_{k+1}}, \quad k \geq 0. \tag{4}$$

*In particular, both surface and volume mean values of $h$ are attained on $D_n$.*

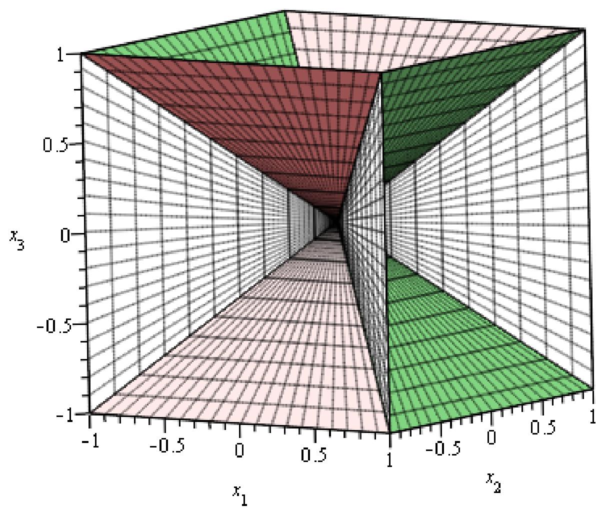


Figure 1: The sets $D_3^{12}(1)$ (white), $D_3^{13}(1)$ (green) and $D_3^{23}(1)$ (coral).

*Proof.* Set

$$M_i := M_i(\boldsymbol{x}) := \max_{j\neq i} |x_j|,$$

and

$$\boldsymbol{x}_t^i := (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n).$$

Using the harmonicity of $h$, we get for $k \geq 1$

$$\begin{aligned}
0 &= \int_{I_n} \Delta h \, d\lambda_n^{\omega_k} = \sum_{i=1}^n \int_{I_n} \omega_k \frac{\partial^2 h}{\partial x_i^2} \, d\lambda_n \\
&= -\sum_{i=1}^n \int_{-r}^r \cdots \int_{-r}^r \frac{\partial \omega_k}{\partial x_i}(\boldsymbol{x}) \frac{\partial h}{\partial x_i}(\boldsymbol{x}) \, dx_i dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \\
&= -\sum_{i=1}^n \int_{-r}^r \cdots \int_{-r}^r \left\{ \left( \int_{-r}^{-M_i} + \int_{M_i}^r \right) \operatorname{sign} x_i \omega_{k-1}(\boldsymbol{x}) \frac{\partial h}{\partial x_i}(\boldsymbol{x}) \, dx_i \right\} \\
&\quad \times dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \\
&= -\sum_{i=1}^n \int_{-r}^r \cdots \int_{-r}^r \left\{ \int_{M_i}^r \omega_{k-1}(\boldsymbol{x}) \frac{\partial}{\partial x_i} [h(\boldsymbol{x}_{-x_i}^i) + h(\boldsymbol{x})] \, dx_i \right\} \\
&\quad \times dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n.
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
0 \;=\; & -\sum_{i=1}^{n}\int_{-r}^{r}\dots\int_{-r}^{r}\{h(\boldsymbol{x}^i_{-r})+h(\boldsymbol{x}^i_{+r}) \qquad (5)\\
& -[h(\boldsymbol{x}^i_{-M_i})+h(\boldsymbol{x}^i_{+M_i})]\}\,dx_1\dots dx_{i-1}dx_{i+1}\dots dx_n
\end{aligned}
$$

if $k=1$ and

$$
\begin{aligned}
0 \;=\; & -\sum_{i=1}^{n}\int_{-r}^{r}\dots\int_{-r}^{r}\int_{M_i}^{r}\omega_{k-2}(\boldsymbol{x})[h(\boldsymbol{x}^i_{-x_i})+h(\boldsymbol{x})]\,dx_i\\
& \times dx_1\dots dx_{i-1}dx_{i+1}\dots dx_n \qquad (6)\\
& +\sum_{i=1}^{n}\int_{-r}^{r}\dots\int_{-r}^{r}\omega_{k-1}(\boldsymbol{x}^i_{+M_i})[h(\boldsymbol{x}^i_{-M_i})+h(\boldsymbol{x}^i_{+M_i})]\\
& \times dx_1\dots dx_{i-1}dx_{i+1}\dots dx_n
\end{aligned}
$$

if $k\geq 2$.
Clearly, (5) is equivalent to (3) and from (6) it follows

$$
0=\int_{I_n}\Delta h\,d\lambda_n^{\omega_k}=\int_{I_n}h\,d\lambda_n^{\omega_{k-2}}-2\int_{D_n}h\,d\lambda_{n-1}^{\omega_{k-1}}, \qquad (7)
$$

which is equivalent to (4). □

Let $M:=M(\boldsymbol{x}):=\max_{1\leq i\leq n}|x_i|$. Analogously to the proof of Theorem 1 (ii), Equation (7) is generalized to:

**Corollary 1.** *If $h\in\mathcal{H}(I_n)$ and $\varphi\in C^2[0,r]$ is such that $\varphi(0)=0$ and $\varphi'(0)=0$, then*

$$
0=\int_{I_n}\varphi(r-M)\Delta h\,d\lambda_n=\int_{I_n}\varphi''(r-M)h\,d\lambda_n-2\int_{D_n}\varphi'(r-M)h\,d\lambda_{n-1}. \qquad (8)
$$

The volume mean-value formula (2) was extended by P. Pizzetti to the following [8, 3, 2].

**The Pizzetti formula.** *If $g\in\mathcal{H}^m(B_n(r))$, then*

$$
\int_{B_n(r)}g\,d\lambda_n=r^n\pi^{n/2}\sum_{k=0}^{m-1}\frac{r^{2k}}{2^{2k}\Gamma(n/2+k+1)}\frac{\Delta^k g(\mathbf{0})}{k!}.
$$

Here we present a similar formula for polyharmonic functions on the hypercube based on integration over the set $D_n$.

**Theorem 2.** *If $g \in \mathcal{H}^m(I_n)$, $m \geq 1$, and $\varphi \in C^{2m}[0,r]$ is such that $\varphi^{(k)}(0) = 0$, $k = 0, 1, \ldots, 2m-1$, then the following identity holds true for any $k \geq 0$:*

$$\int_{I_n} \varphi^{(2m)}(r-M) g \, d\lambda_n = 2 \sum_{s=0}^{m-1} \int_{D_n} \varphi^{(2s+1)}(r-M) \Delta^{m-s-1} g \, d\lambda_{n-1}, \tag{9}$$

*where $\varphi^{(j)}(t) = \frac{d^j \varphi}{dt^j}(t)$.*

*Proof.* Equation (9) is a direct consequence from (8):

$$\begin{aligned} 0 &= \int_{I_n} \varphi(r-M) \Delta^m g \, d\lambda_n \\ &= -2 \int_{D^n} \varphi^{(1)}(r-M) \Delta^{m-1} g \, d\lambda_{n-1} + \int_{I_n} \varphi^{(2)}(r-M) \Delta^{m-1} g \, d\lambda_n \\ &= \ldots = -2 \sum_{s=0}^{m-1} \int_{D_n} \varphi^{(2s+1)} \Delta^{m-s-1} g \, d\lambda_{n-1} + \int_{I_n} \varphi^{(2m)} g \, d\lambda_n. \end{aligned}$$

□

## 3. A relation to best one-sided $L^1$-approximation by harmonic functions

Theorem 1 suggests that for a certain positive cone in $C(I_n)$ the set $D_n$ is a characteristic set for the best one-sided $L^1$-approximation with respect to $\mathcal{H}(I_n)$ as it is explained and illustrated by the examples presented below.

For a given $f \in C(I_n)$, let us introduce the following subset of $\mathcal{H}(I_n)$:

$$\mathcal{H}_-(I_n, f) := \{h \in \mathcal{H}(I_n) \mid h \leq f \text{ on } I_n\}.$$

A harmonic function $h_*^f \in \mathcal{H}_-(I_n, f)$ is said to be a best one-sided $L^1$-approximant from below to $f$ with respect to $\mathcal{H}(I_n)$ if

$$\|f - h_*^f\|_1 \leq \|f - h\|_1 \text{ for every } h \in \mathcal{H}_-(I_n, f),$$

where

$$\|g\|_1 := \int_{I_n} |g| \, d\lambda_n.$$

Theorem 1 (ii) readily implies the following ([1, 7]).

**Theorem 3.** *Let $f \in C(I_n)$ and $h_*^f \in \mathcal{H}_-(I_n, f)$. Assume further that the set $D_n$ belongs to the zero set of the function $f - h_*^f$. Then $h_*^f$ is a best one-sided $L^1$-approximant from below to $f$ with respect to $\mathcal{H}(I_n)$.*

**Corollary 2.** *If $f \in C^1(I_n)$, any solution $h$ of the problem*

$$h_{|D_n} = f_{|D_n}, \quad \nabla h_{|D_n} = \nabla f_{|D_n}, \quad h \in \mathcal{H}_-(I_n, f), \tag{10}$$

*is a best one-sided $L^1$-approximant from below to $f$ with respect to $\mathcal{H}(I_n)$.*

**Corollary 3.** *If $f(\boldsymbol{x}) = g(\boldsymbol{x}) \prod_{1\le i<j\le n}(x_i^2 - x_j^2)^2$, where $g \in C(I_n)$ and $g \ge 0$ on $I_n$, then $h_*^f(\boldsymbol{x}) \equiv 0$ is a best one-sided $L^1$-approximant from below to $f$ with respect to $\mathcal{H}(I_n)$.*

**Example 1.** Let $n = 2$, $r = 1$ and $f_1(x_1, x_2) = x_1^2 x_2^2$. By Corollary 2, the solution $h_*^{f_1}(x_1, x_2) = -x_1^4/4 + \frac{3}{2}x_1^2x_2^2 - x_2^4/4$ of the interpolation problem (10) with $f = f_1$ is a best one-sided $L^1$-approximant from below to $f_1$ with respect to $\mathcal{H}(I_2)$ and $\|f_1 - h_*^{f_1}\|_1 = 8/45$. Since the function $f_1$ belongs to the positive cone of the partial differential operator $\mathcal{D}_{2,2}^4 := \frac{\partial^4}{\partial x_1^2 \partial x_2^2}$ (that is, $\mathcal{D}_{2,2}^4 f_1 > 0$), one can compare the best harmonic one-sided $L^1$-approximation to $f_1$ with the corresponding approximation from the linear subspace of $C(I_2)$:

$$\mathcal{B}^{2,2}(I_2) := \{b \in C(I_2) \mid b(x_1, x_2) = \sum_{j=0}^{1}[a_{0j}(x_1)x_2^j + a_{1j}(x_2)x_1^j]\}.$$

The possibility for explicit constructions of best one-sided $L^1$-approximants from $\mathcal{B}^{2,2}(I_2)$, is studied in [4]. The functions $f_1 - b_*^{f_1}$ and $f_1 - b_{f_1}^*$, where $b_*^{f_1}$ and $b_{f_1}^*$ are the unique best one-sided $L^1$-approximants to $f_1$ with respect to $\mathcal{B}^{2,2}(I_2)$ from below and above, respectively, play the role of basic error functions of the canonical one-sided $L^1$-approximation by elements of $\mathcal{B}^{2,2}(I_2)$. For instance, $b_*^{f_1}$ can be constructed as the unique interpolant to $f_1$ on the boundary $\Diamond := \{(x_1, x_2) \in I_2 \mid |x_1| + |x_2| = 1\}$ of the inscribed square and $\|f_1 - b_*^{f_1}\|_1 = 14/45$ (Fig. 2).

**Example 2.** Let $n = 2$, $r = 1$ and $f_2(x_1, x_2) = x_1^8 + 14x_1^4x_2^4 + x_2^8$. The solution $h_*^{f_2}(x_1, x_2) = x_1^8 + x_2^8 - 28(x_1^6x_2^2 + x_1^2x_2^6) + 70x_1^4x_2^4$ of (10) with $f = f_2$ is a best one-sided $L^1$-approximant from below to $f_2$ with respect to $\mathcal{H}(I_2)$ and $\|f_2 - h_*^{f_2}\| = 8/75$. It can also be verified that $\|f_2 - b_*^{f_2}\| = 121/900$ (see Fig. 3).

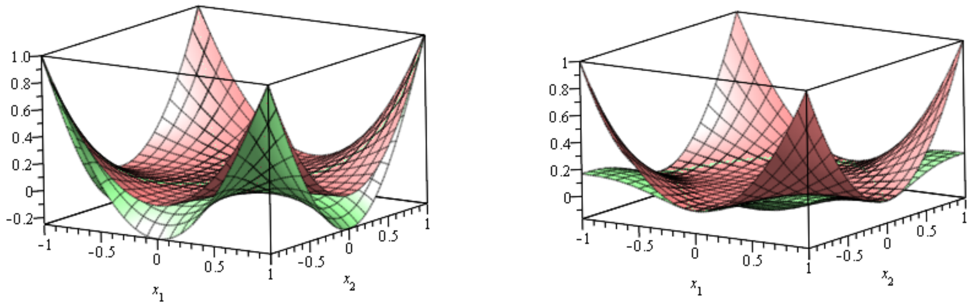


Figure 2: The graphs of the function $f_1(x_1, x_2) = x_1^2 x_2^2$ (coral) and its best one-sided $L^1$-approximants from below, $h_*^{f_1}$ with respect to $\mathcal{H}(I_2)$ (left) and $b_*^{f_1}$ with respect to $\mathcal{B}^{2,2}(I_2)$ (right).

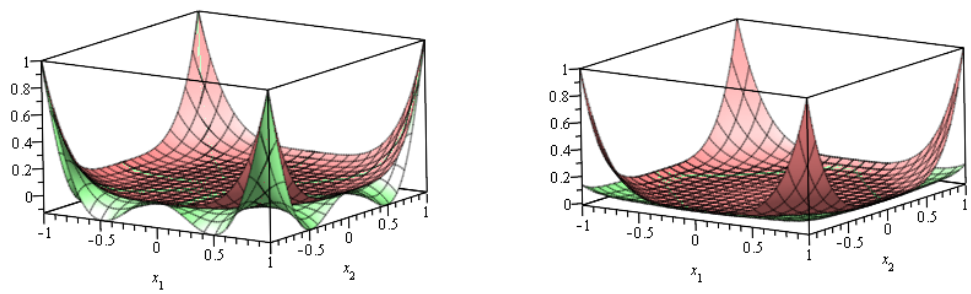


Figure 3: The graphs of the function $f_2(x_1, x_2) = x_1^8 + 14x_1^4 x_2^4 + x_2^8$ (coral) and its best one-sided $L^1$-approximants from below, $h_*^{f_2}$ with respect to $\mathcal{H}(I_2)$ (left) and $b_*^{f_2}$ with respect to $\mathcal{B}^{2,2}(I_2)$ (right).

**Remark 1.** Let $\varphi \in C^2[0, r]$ is such that $\varphi(0) = 0$, $\varphi'(0) = 0$, and $\varphi' \geq 0$, $\varphi'' \geq 0$ on $[0, r]$. It follows from (8) that Theorem 3 also holds for the best weighted $L^1$-approximation from below with respect to $\mathcal{H}(I_n)$ with weight $\varphi''(r - M)$. The smoothness requirements were used for brevity and wherever possible they can be weakened in a natural way.

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